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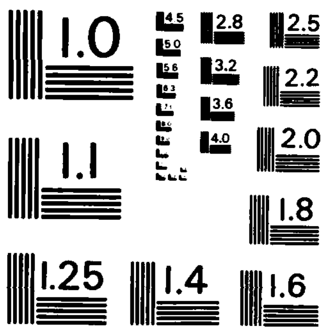

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0. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This ~~The present~~ paper considers a multi-compartment storage model with one way flow. The inputs and outputs for each compartment are controlled by a denumerable state Markov chain. Assuming finite first and second moment conditions, the limit behavior of the compartments are examined. It is shown that the diverging compartments under suitable normalization converge to functionals of multivariate Brownian motion, independent of those compartments which converge without normalization.

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by

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On the Limit Behavior of a Multi-Compartment Storage Model With an Underlying Markov Chain II: With Normalization

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Abstract

The present paper considers a multi-compartment storage model with one way flow. The inputs and outputs for each compartment are controlled by a denumerable state Markov chain. Assuming finite first and second moment conditions, the limit behavior of the compartments are examined. It is shown that the diverging compartments under suitable normalization converge to functionals of Brownian motion, independent of those compartments which converge without normalization.

1. INTRODUCTION

In this paper, a multi-compartment storage model with one-way flow is considered. This model is a generalization of a one compartment model considered by Puri [8], Puri and Senturia [9], [10], Balagopal [1], Puri and Woolford [11], and others. In Tollar [12], it was established that under first moment criteria, as time increases, the subcritical compartments converge in distribution, while the critical and supercritical compartments diverge. This paper examines the limit behavior of the divergent compartments when suitably normalized.

In section 2, the model being considered is described, and some results in [12] are summarized. In section 3, intermediate results for maximums of processes defined on a Markov are obtained. In particular, these results show that the normalized difference between the process considered and the similar maximum defined on i.i.d. random variables converges in probability to zero. In section 4, it is shown that the divergent compartments appropriately normalized converge to functionals of Brownian motion and that this behavior is independent of the convergent compartments.

2. THE MODEL

Let $\{X_n;\ n = 0, 1, 2, \dots\}$ be an aperiodic, recurrent, irreducible, Markov chain with denumerable state space J and stationary measure $\underset{\sim}{\pi}$. For $i \in J$, let $\{\underset{\sim}{V}_n(i) = (V_{0,n}(i), \dots, V_{k,n}(i)):\ n = 1, 2, \dots\}$ be a sequence of i.i.d. $k+1$-tuples, independent of $\{X_n\}$ and of $\{\underset{\sim}{V}_n(j)\}$ for $j \neq i$. We then consider a model in which $\underset{\sim}{Z}(n) = (Z_1(n), \dots, Z_k(n))$ represents the amount of material at time n in the k compartments. For each compartment ℓ, $Z_\ell(n)$ is given by

$$Z_\ell(n) = \min_{0 \le j_1 \le \ldots \le j_{\ell-1} \le n} [S_0(j_1) + (S_1(j_2) - S_1(j_1)) + \ldots$$

$$+ (S_{\ell-1}(n) - S_{\ell-1}(j_{\ell-1}))] \tag{2.1}$$

$$- \min_{0 \le j_1 \le \ldots \le j_\ell \le n} [S_0(j_1) + \ldots + (S_\ell(n) - S_\ell(j_\ell))]$$

where $S_i(m)$ is defined by

$$S_i(m) = \sum_{j=1}^{m} V_{i,j}(X_j). \tag{2.2}$$

For further discussion of the motivation and origin of (2.1), see Tollar [12] .

For $0 \le \ell \le k$, we define $E_\pi V_\ell$ by

$$E_\pi V_\ell = \sum_{j \in J} \pi_j \cdot E[V_{\ell,1}(j)] . \tag{2.3}$$

We will refer to compartment ℓ as either subcritical, critical or supercritical when $E_\pi V_\ell - \min_{0 \le i < \ell} (E_\pi V_i)$ is greater than, equal to, or less than 0, respectively. It is established in [12] that as n tends to infinity, the subcritical compartments converge, while the critical and super critical compartments diverge.

The convergence properties of the compartments were shown to not depend on the initial distributions of the compartments. Further, for $\{\hat{X}_n : n = 0, 1, \ldots\}$ the dual Markov chain (for definition, see Çinlar [4]), it was shown the limit behavior of $\underset{\sim}{Z}(n)$ coincides with the behavior of $\hat{\underset{\sim}{Z}}(n) = (\hat{Z}_1(n), \ldots, \hat{Z}_k(n))$, where $\hat{Z}_i(n)$ is defined as follows:

$$\hat{Z}_i(n) = \min_{0 \le j_1 \le \dots \le j_{i-1} \le n} (\hat{S}_{i-1}(j_1) + [\hat{S}_{i-2}(j_2) - \hat{S}_{i-2}(j_1)] + \dots + [\hat{S}_0(n) - \hat{S}_0(j_{i-1})])$$

$$- \min_{0 \le j_1 \le \dots \le j_i \le n} (\hat{S}_i(j_1) + \dots + [\hat{S}_0(n) - \hat{S}_0(j_i)]), \tag{2.4}$$

where $\hat{S}_j(m)$ is defined as

$$\hat{S}_j(m) = \sum_{\ell=1}^{m} V_{j,\ell}(\hat{X}_\ell). \tag{2.5}$$

It was also established in [12], that the initial distribution of X_0 was not related to the limit behavior of the compartments. As such, we will set $\underset{\sim}{Z}(0) \equiv 0$, and $X_0 \equiv \underset{\sim}{\pi}$, and use either expression (2.1) or (2.4), whichever proves more useful.

3. INTERMEDIATE ASYMPTOTIC RESULTS

When working with processes with underlying Markov chains, the typical approach (see Chung [3]) is to use the return times to an arbitrary state $i_0 \in J$ to generate a sequence of i.i.d. random variables. It is shown that functionals of these i.i.d. random variables have the same asymptotic behavior as the processes under consideration. This is the technique to be used on certain functionals related to the process $\underset{\sim}{Z}(n)$ in this section. For arbitrary $i_0 \in J$, we define $t_n(i_0)$ recursively by

$$t_n(i_0) = \min\{j > t_{n-1}(i_0) : X_j = i_0\}, \qquad t_0(i_0) \equiv 0. \tag{3.1}$$

Let $\{Y_n(i), n = 1, 2, \dots\}$ be i.i.d. sequences defined for all $i \in J$, independent of $\{Y_n(j), n = 1, 2, \dots\}$ $i \ne j$, and also of $\{X_n\}$. If $E_\pi|Y| < \infty$, it can be shown (see Balagopal [1]) that

$$Y_n^*(i_0) = [\sum_{i=t_{n-1}(i_0)+1}^{t_n(i_0)} Y_i(X_i)] \tag{3.2}$$

is an i.i.d. sequence with $E[Y_n^*(i_0) - (t_n(i_0) - t_{n-1}(i_0))E_\pi Y] = 0$, for $n > 1$. If we let $\sigma_{i_0}^2 = E[Y_1^*(i_0)^2 | X_0 = i_0]$, we have $\sigma_{i_0}^2 < \infty$ if and only if $\sigma_j^2 < \infty$ for all $j \in \mathcal{J}$, in which case $\pi_j \sigma_j^2 = \pi_{i_0} \sigma_{i_0}^2$. We will also define

$$\tilde{Y}_n(i_0) = \sum_{i=t_{n-1}(i_0)+1}^{t_n(i_0)} |Y_i(X_i)|. \tag{3.3}$$

In most cases the state i_0 is understood, and i_0 will be dropped in expression (3.1), (3.2), and (3.3). Let $\{(Y_{1,n}(i), \ldots, Y_{\ell,n}(i)) : n = 1, 2, \ldots\}$ be an i.i.d. ℓ-tuple sequence, independent of $\{Y_{1,n}(j), \ldots, Y_{\ell,n}(j)) : n = 1, 2, \ldots\}$, $i \neq j$, and also of $\{X_n\}$, the Markov chain with stationary measure $\underline{\pi}$. Further, let us define $M_{i_0}(n)$ by

$$M_{i_0}(n) = \sum_{i=1}^{n} I(X_i = i_0) . \tag{3.4}$$

We then have the following intermediate theorem.

<u>THEOREM 3.1</u>: <u>Let</u> $E_\pi |Y_i| < \infty$, <u>and</u> $E(\tilde{Y}_i^2 | X_0 = i_0) < \infty$, $1 \le i \le \ell$. <u>Then as</u> $n \to \infty$

$$n^{-1/2}[\max_{0 \le j_1 \le \ldots \le j_\ell \le n} (\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i))$$

$$- \max_{0 \le j_1 \le \ldots \le j_\ell \le M_{i_0}(n)} (\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}^*)] \xrightarrow{P} 0 .$$

PROOF. It can easily be seen that

$$\Big| \max_{0 \le j_1 \le \dots \le j_\ell \le n} \Big(\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \Big) \tag{3.5}$$

$$- \max_{t_1 \le j_1 \le \dots \le j_\ell \le t_{M_{i_0}}(n)} \Big(\sum_{i=t_1+1}^{j_1} Y_{1,i}(X_i) + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \Big) \Big|$$

$$\le \sum_{j=1}^{\ell} \sum_{i=1}^{t_1} | Y_{j,i}(X_i) | + \sum_{j=1}^{\ell} \sum_{i=t_{M_{i_0}}(n)+1}^{n} | Y_{j,i}(X_i) | .$$

Since $P(t_1 < \infty) = 1$, we have that $n^{-1/2} \sum_{j=1}^{\ell} \sum_{i=1}^{t_1} | Y_{j,i}(X_i) | \xrightarrow{p} 0$.

In (Chung [3]), it is shown under the assumption that $E_\pi |Y_j| < \infty$, that

$$n^{-1/2} \sum_{i=t_{M_{i_0}}(n)+1}^{n} |Y_{j,i}(X_i)| \xrightarrow{p} 0.$$

Thus, we get from (3.5) that

$$n^{-1/2} \Big| \max_{0 \le j_1 \le \dots \le j_\ell \le n} \Big(\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \Big) \tag{3.6}$$

$$- \max_{t_1 \le j_1 \le \dots \le j_\ell \le t_{M_{i_0}}(n)} \Big(\sum_{i=t_1+1}^{j_1} Y_{1,i}(X_i) + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \Big) \Big| \xrightarrow{p} 0.$$

Let us define $t_{M_{i_0}(n)+1} \equiv t_{M_{i_0}}(n) + 1$, $r_0 \equiv 1$, and for $1 \le r_1 \le r_2 \le \dots \le r_\ell \le M_{i_0}(n)$

$$B(r_1, r_2, \ldots, r_\ell) = \{(j_1, \ldots, j_\ell):\ j_1 \le j_2 \le \ldots \le j_\ell ,$$

$$t_{r_1} \le j_1 < t_{r_1+1},\ t_{r_2} \le j_2 < t_{r_2+1}, \ldots, t_{r_\ell} \le j_\ell < t_{r_\ell+1} \} , \tag{3.7}$$

$$U(r_1, \ldots, r_\ell) = \max_{(j_1, \ldots, j_\ell) \in B(r_1, \ldots, r_\ell)} [\sum_{k=1}^{\ell-1} \sum_{i=t_{r_k}+1}^{j_k} (Y_{k,i}(X_i)$$

$$- Y_{k+1,i}(X_i)) + \sum_{i=t_{r_\ell}+1}^{j_\ell} Y_{\ell,i}(X_i)] . \tag{3.8}$$

Then we have

$$\Big| \max_{t_1 \le j_1 \le \ldots \le j_\ell \le t_{M_{i_\ell}(n)}} \Big(\sum_{i=t_1+1}^{j_1} Y_{1,i}(X_i) + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \Big)$$

$$- \max_{1 \le r_1 \le \ldots \le r_\ell \le M_{i_0}(n)} \Big(\sum_{k=1}^{\ell} \sum_{i=t_{r_{k-1}}+1}^{t_{r_k}} Y_{k,i}(X_i) \Big) \Big|$$

$$\le \max_{1 \le r_1 \le \ldots \le r_\ell \le M_{i_0}(n)} (U(r_1, \ldots, r_\ell)) \tag{3.9}$$

$$\le \sum_{k=1}^{\ell-1} \max_{1 \le j < M_{i_0}(n)} \Big(\sum_{i=t_j+1}^{t_{j+1}} |Y_{k,i}(X_i) - Y_{k+1,i}(X_i)|$$

$$+ \max_{1 \le j < M_{i_0}(n)} \Big(\sum_{i=t_j+1}^{t_{j+1}} |Y_{\ell,i}(X_i)| \Big).$$

Since we have

$$
\begin{aligned}
&P\Big(\max_{1 \le j < M_{i_0}(n)} \Big(\sum_{i=t_j+1}^{t_{j+1}} |Y_{\ell,i}(X_i)|\Big) > \varepsilon\sqrt{n}\Big) \\
&\le P\Big(\bigcup_{j=1}^{n} \Big\{\sum_{i=t_j+1}^{t_{j+1}} |Y_{\ell,i}(X_i)| > \varepsilon\sqrt{n}\Big\}\Big) \qquad (3.10) \\
&\le n\, P\Big(\sum_{i=t_1+1}^{t_2} |Y_{\ell,i}(X_i)| > \varepsilon\sqrt{n}\Big),
\end{aligned}
$$

and $E(\tilde{Y}^2_{\ell,1} | X_0 = i_0) < \infty$, we get that

$$\lim_{n\to\infty} n\, P\Big(\sum_{i=t_1+1}^{t_2} |Y_{\ell,i}(X_i)| > \varepsilon\sqrt{n}\Big) = 0, \qquad (3.12)$$

with a similar result for the other term in (3.9). Thus we get

$$
\begin{aligned}
&n^{-1/2}\Big[\max_{0 \le j_1 \le \dots \le j_\ell \le n} \Big(\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i)\Big) \\
&- \max_{1 \le r_1 \le \dots \le r_\ell \le M_{i_0}(n)} \Big(\sum_{k=1}^{\ell} \sum_{i=t_{r_{k-1}}+1}^{t_{r_k}} Y_{k,i}(X_i)\Big)\Big] \xrightarrow{p} 0. \qquad (3.13)
\end{aligned}
$$

and the theorem is established. □

Before continuing with results on the maximums of sequences defined on Markov chains, we establish the following lemma about i.i.d. random variables with finite variance and means less than zero.

LEMMA 3.2: Let $\{X_n : n = 1, 2, \dots\}$ be an i.i.d. sequence with $EX < 0$, $EX^2 < \infty$. Then, as $n \to \infty$, $n^{-1/2} \max_{0 \le j \le k \le n} \left(\sum_{i=j+1}^{k} X_i \right) \xrightarrow{p} 0$.

PROOF. Set $S_j = \sum_{i=1}^{j} X_i$,

$$R_n = \max_{0 \le j \le k \le n} \left(\sum_{i=j+1}^{k} X_i \right) = \max_{0 \le j \le k \le n} (S_k - S_j).$$

For any m, divide [0,n] into $[m\sqrt{n} + 1]$ approximately equal sized parts, setting $n_i = [i\sqrt{n}\, m^{-1}]$ with $n_{[m\sqrt{n}+1]} \equiv n$, where [x] is the greatest integer of x. Then let

$$i_\ell = \min\{i : n_i \ge \ell\} \qquad \text{and} \tag{3.14}$$

$$R_{n,m} = \max_{0 \le j \le n} \left[\max\left(0, \max_{i_j \le k \le [m\sqrt{n}+1]} (S_{n_k} - S_j)\right)\right] \tag{3.15}$$

Clearly, for all n,m,

$$R_n \ge R_{n,m} \,. \tag{3.16}$$

To establish that as $n, m \to \infty$, $n^{-1/2}(R_n - R_{n,m}) \xrightarrow{p} 0$, we define

$$E_{k,n} = \{\omega : \bigcap_{\ell=1}^{k} [\max_{0 \le j < \ell} (S_\ell - S_j) \le \varepsilon\sqrt{n}\,],$$

$$\max_{0 \le j \le k} (S_k - S_j) > \varepsilon\sqrt{n}\,\}, \tag{3.17}$$

$$\begin{cases} E_{k,n,1} = E_{k,n} \cap \{\omega : S_{n_{i_k}} - S_k \ge -\varepsilon\sqrt{n}/2 \\ \\ E_{k,n,2} = E_{k,n} \cap \{\omega : S_{n_{i_k}} - S_k < -\varepsilon\sqrt{n}/2 \end{cases} \tag{3.18}$$

From (3.17) and (3.18) it can be seen that

$$P(R_n > \varepsilon\sqrt{n}\ ) = \sum_{k=1}^{n} P(E_{k,n}) = \sum_{k=1}^{n} P(E_{k,n,1}) + \sum_{k=1}^{n} P(E_{k,n,2}) . \qquad (3.19)$$

Also, from (3.15) we have

$$\begin{aligned} P(E_{k,n,1}) &\le P(E_{k,n}, \max_{0\le j\le k} (S_{n_{i_k}} - S_j) > \varepsilon\sqrt{n}/2\ ) \\ &\le P(E_{k,n}, R_{n,m} > \varepsilon\sqrt{n}/2\ ) . \end{aligned} \qquad (3.20)$$

Thus, we have

$$\sum_{k=1}^{n} P(E_{k,n,1}) \le P(R_{n,m} > \varepsilon\sqrt{n}/2\ ) . \qquad (3.21)$$

Since $S_{n_{i_k}} - S_k$ is independent of $E_{k,n}$, we have

$$P(E_{k,n,2}) = P(E_{k,n})\ P(S_{n_{i_k}} - S_k < -\varepsilon\sqrt{n}/2\ ) . \qquad (3.22)$$

Since the X's are i.i.d. we get

$$\begin{aligned} P(S_{n_{i_k}} - S_k < -\varepsilon\sqrt{n}/2\ ) &= P(S_{n_{i_k} - k} < -\varepsilon\sqrt{n}/2\ ) \\ &= P(\sum_{i=1}^{n_{i_k}-k} (X_i - EX) < -\varepsilon\sqrt{n}/2 - (n_{i_k} - k)EX) . \end{aligned} \qquad (3.23)$$

As can be easily established, $n_{i_k} - k \le \sqrt{n}\, m^{-1} + 1$. Since $EX < 0$,

$$P(S_{n_{i_k}} - S_k < -\varepsilon\sqrt{n}/2\ ) < P(\sum_{i=1}^{n_{i_k}-k} (X_i - EX) < -\sqrt{n}\ (\varepsilon/2 + (m^{-1} + n^{-1/2})EX))) . \qquad (3.24)$$

For any ε, choose m and n sufficiently large so that $\varepsilon/2 + (m^{-1} + n^{-1/2})EX > 0$. Then from Chebyshev's inequality, we get

$$P(S_{n_{i_k}} - S_k < -\varepsilon\sqrt{n}/2) < (n_{i_k} - k)\sigma^2 n^{-1}(\varepsilon/2 + (m^{-1} + n^{-1/2})EX)^{-2}$$
$$\le (\sqrt{n}\, m^{-1} + 1)\sigma^2 n^{-1}(\varepsilon/2 + (m^{-1} + n^{-1/2})EX)^{-2} . \tag{3.25}$$

Therefore

$$\sum_{k=1}^{n} P(E_{k,n,2}) \le \sum_{k=1}^{n} P(E_{k,n})(\sqrt{n}\, m^{-1} + 1)\sigma^2 n^{-1}(\varepsilon/2 + (m^{-1} + n^{-1/2})EX)^{-2}$$
$$< (\sqrt{n}\, m^{-1} + 1)\sigma^2 n^{-1}(\varepsilon/2 + (m^{-1} + n^{-1/2})EX)^{-2} \tag{3.26}$$

Thus, for all $m > -2EX/\varepsilon$, as $n \to \infty$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} P(E_{k,n,2}) = 0. \tag{3.27}$$

Combining (3.16), (3.19), (3.21) and (3.27), we have for $m > -2EX/\varepsilon$, there is an N where for all $n > N$,

$$P(R_{n,m} > \varepsilon\sqrt{n}) \le P(R_n > \varepsilon\sqrt{n}) < P(R_{n,m} > \varepsilon\sqrt{n}/2\ ) + \varepsilon . \tag{3.28}$$

Clearly, from (3.15) we have

$$R_{n,m} = \max_{1 \le k \le [m\sqrt{n} + 1]} \left(\max_{0 \le j \le n_k} (S_{n_k} - S_j) \right). \tag{3.29}$$

Thus,

$$P(R_{n,m} > \delta\sqrt{n}) \le \sum_{k=1}^{[m\sqrt{n}+1]} P(\max_{0\le j\le n_k} (S_{n_k} - S_j) > \delta\sqrt{n})$$

$$= \sum_{k=1}^{[m\sqrt{n}+1]} P(\max_{0\le j\le n_k} (S_j) > \delta\sqrt{n}), \tag{3.30}$$

since $\max_{0\le j\le n_k} (S_{n_k} - S_j) \overset{d}{=} \max_{0\le j\le n_k} (S_j)$. As such,

$$P(R_{n,m} > \delta\sqrt{n}) \le [m\sqrt{n}+1]\, P(\sup_{j\ge 0} (S_j) > \delta\sqrt{n}) \tag{3.31}$$

By (Kiefer and Wolfowitz [7]), since $EX^2 < \infty$, we have $E \sup_{j\ge 0} (S_j) < \infty$. Thus, by the well known result, $\lim_{x\to\infty} xP(X > x) = 0$ if $EX^+ < \infty$, we get

$\lim_{n\to\infty} [m\sqrt{n}+1]\, P(\sup_{j\ge 0} (S_j) > \delta\sqrt{n}) = 0$, for all δ. Thus, we have $P(R_{n,m} > \delta\sqrt{n}) \to 0$,

which yields from (3.28) $P(R_n > \varepsilon\sqrt{n}) \to 0$. □

In the following theorem, define $(\ell_1, \ell_2, \ldots, \ell_m)$ as the indices of $E_\pi Y_1, \ldots, E_\pi Y_\ell$ such that $E_\pi Y_{\ell_j} = 0$.

<u>THEOREM 3.3</u>: <u>Let</u> $E_\pi |Y_i| < \infty$, $E_\pi Y_i \le 0$, and $E(\tilde{Y}_i^2 | X_0 = i_0) < \infty$, $1 \le i \le \ell$. <u>Then</u>, <u>as</u> $n \to \infty$,

$$n^{-1/2}\Big[\max_{0\le j_1\le \ldots \le j_\ell \le n} \Big(\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i)\Big)$$

$$- \max_{0\le j_1\le \ldots\le j_m \le [\pi_{i_0} n]} \Big(\sum_{k=1}^{m} \sum_{i=j_{k-1}+1}^{j_k} Y^*_{\ell_k,i}\Big)\Big] \xrightarrow{P} 0.$$

PROOF: From Theorem 3.1, we know that we need only show that

$$n^{-1/2}\Big[\max_{0\le j_1\le\dots\le j_\ell\le M_{i_0}(n)}\Big(\sum_{k=1}^{\ell}\sum_{i=j_{k-1}+1}^{j_k} Y^*_{k,i}\Big)$$

$$-\max_{0\le j_1\le\dots\le j_m\le[\pi_{i_0}n]}\Big(\sum_{k=1}^{m}\sum_{i=j_{k-1}+1}^{j_k} Y^*_{\ell_k,i}\Big)\Big]\xrightarrow{P}0. \tag{3.32}$$

Since $n^{-1}M_{i_0}(n)\to\pi_{i_0}$, there is a decreasing sequence $\{\varepsilon_n: n=1,2,\dots\}$ where $\varepsilon_n\downarrow 0$ and for all n,

$$P([(\pi_{i_0}-\varepsilon_n)n]\le M_{i_0}(n)\le[(\pi_{i_0}+\varepsilon_n)n])>1-\varepsilon_n. \tag{3.33}$$

Let us define $\ell_1(n)=[(\pi_{i_0}-\varepsilon_n)n]$, $\ell(n)=[\pi_{i_0}n]$, $\ell_2(n)=[(\pi_{i_0}+\varepsilon_n)n]$.

Then, for any $\varepsilon>0$,

$$P\Big(\Big|\max_{1\le j_1\le\dots\le j_\ell\le M_{i_0}(n)}\Big(\sum_{k=1}^{\ell}\sum_{i=j_{k-1}+1}^{j_k} Y^*_{k,i}\Big)$$

$$-\max_{1\le j_1\le\dots\le j_\ell\le \ell(n)}\Big(\sum_{k=1}^{\ell}\sum_{i=j_{k-1}+1}^{j_k} Y^*_{k,i}\Big)\Big|>\varepsilon\sqrt{n}\Big)$$

$$\le P\Big(\max_{1\le j_1\le\dots\le j_\ell\le \ell_2(n)}\Big(\sum_{k=1}^{\ell}\sum_{i=j_{k-1}}^{j_k} Y^*_{k,i}\Big)$$

$$-\max_{1\le j_1\le\dots\le j_\ell\le \ell_1(n)}\Big(\sum_{k=1}^{\ell}\sum_{i=j_{k-1}}^{j_k} Y^*_{k,i}\Big)>\varepsilon\sqrt{n}\Big)+\varepsilon_n \tag{3.34}$$

$$\le P\Big(\max_{\ell_1(n)\le j_1\le\dots\le j_\ell\le \ell_2(n)}\Big(\sum_{i=\ell_1(n)+1}^{j_1} Y^*_{1,i}$$

$$+\sum_{k=2}^{\ell}\sum_{i=j_{k-1}+1}^{j_k} Y^*_{k,i}\Big)>\varepsilon\sqrt{n}\Big)+\varepsilon_n .$$

Because the $Y^*_{\ell,i}$'s are i.i.d, we have

$$\max_{\ell_1(n) \le j_1 \le \ldots \le j_2 \le \ell_2(n)} \left(\sum_{i=\ell_1(n)+1}^{j_i} Y^*_{1,i} + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y^*_{k,i} \right)$$

$$\stackrel{d}{=} \max_{0 \le j_1 \le \ldots \le j_\ell \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{j_1-1} Y^*_{1,i} + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}}^{j_k-1} Y^*_{k,i} \right). \tag{3.35}$$

Now, we have

$$\max_{0 \le j_1 \le \ldots \le j_m \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{j_1-1} Y^*_{\ell_1,i} + \sum_{k=2}^{m} \sum_{i=j_{k-1}}^{j_k-1} Y^*_{\ell_k,i} \right)$$

$$\le \max_{0 \le j_1 \le \ldots \le j_\ell \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{j_1-1} Y^*_{1,i} + \sum_{k=2}^{\ell} \sum_{i=j_{k-1}}^{j_k-1} Y^*_{k,i} \right),$$

$$\le \max_{0 \le j_1 \le \ldots \le j_m \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{j_1-1} Y^*_{\ell_1,i} + \sum_{k=2}^{m} \sum_{i=j_{k-1}}^{j_k-1} Y^*_{\ell_k,i} \right) \tag{3.36}$$

$$+ \sum_{k=1}^{m} \sum_{j=\ell_{k-1}+1}^{\ell_k-1} \left[\max_{0 \le r \le s \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=r}^{s-1} (Y^*_{j,i} - Y^*_{\ell_k,i}) \right) \right]$$

$$+ \sum_{k=\ell_m+1}^{\ell} \max_{0 \le r \le s \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=r}^{s-1} Y^*_{k,i} \right).$$

Since $E(Y^*_{j,i} - Y^*_{\ell_k,i}) < 0$ for $\ell_{k-1} < j < \ell_k$, we have from lemma 3.2 that

$$n^{-1/2} \max_{0 \le r \le s \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=r}^{s-1} Y^*_{j,i} - Y^*_{\ell_k,i} \right) \xrightarrow{P} 0, \tag{3.37}$$

and similarly $n^{-1/2} \max\limits_{0 \le r \le s \le \ell_2(n) - \ell_1(n)} \left(\sum\limits_{i=r}^{s-1} Y^*_{j,i} \right) \xrightarrow{p} 0$ for $j > \ell_k$.

In addition, since

$$\max_{0 \le j_1 \le \ldots \le j_m \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{j_1-1} Y^*_{\ell_1,i} + \sum_{k=2}^{m} \sum_{i=j_{k-1}}^{j_k-1} Y^*_{\ell_k,i} \right)$$

$$= \max_{0 \le r_1 \le \ldots \le r_m \le \ell_2(n) - \ell_1(n)} \left(\sum_{j=1}^{m-1} \sum_{i=0}^{r_j-1} (Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i}) + \sum_{i=0}^{r_m-1} Y^*_{\ell_m,i} \right), \tag{3.38}$$

we get

$$P\left(\max_{0 \le r_1 \le \ldots \le r_m \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{r_1-1} Y^*_{\ell_1,i} + \sum_{k=2}^{m} \sum_{i=r_{k-1}}^{r_k-1} Y^*_{\ell_k,i} \right) > \varepsilon\sqrt{n} \right)$$

$$\le \sum_{j=1}^{m-1} P\left(\max_{0 \le r \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{r-1} Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i} \right) > \varepsilon\sqrt{n}/m \right) \tag{3.39}$$

$$+ \quad P\left(\max_{0 \le r \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{r-1} Y^*_{\ell_m,i} \right) > \varepsilon\sqrt{n}/m \right).$$

For $1 \le j < m$, we have $E(Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i}) = 0$, and

$$P\left(\max_{0 \le r \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{r-1} (Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i}) \right) > \frac{\varepsilon}{m} \frac{n^{1/2}}{(\ell_2(n) - \ell_1(n))^{1/2}} (\ell_2(n) - \ell_1(n))^{1/2} \right) \tag{3.40}$$

$$\le P\left(\max_{0 \le r \le \ell_2(n) - \ell_1(n)} \left(\sum_{i=0}^{r-1} (Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i}) \right) > \frac{\varepsilon}{m} (2\varepsilon_n + n^{-1})^{-1/2} (\ell_2(n) - \ell_1(n))^{1/2} \right).$$

Thus, since $\max\limits_{0 \le r \le n} \left(\sum\limits_{i=0}^{r-1} Y^*_{\ell_j,i} - Y^*_{\ell_{j+1},i} \right)$ converges to an absolute normal, (see Erdös and Kac [6]), we have that since $(2\varepsilon_n + n^{-1})^{-1/2} \to \infty$ that expression (3.40) converges to 0, regardless of whether or not $\ell_2(n) - \ell_1(n)$ tends to infinity. As such, from (3.34), (3.36), (3.37) and (3.39), we have that

$$
\begin{aligned}
n^{-1/2}\Big[& \max_{0 \le j_1 \le \dots \le j_\ell \le n} \Big(\sum_{k=1}^{\ell} \sum_{i=j_{k-1}+1}^{j_k} Y_{k,i}(X_i) \\
& - \max_{1 \le r_1 \le \dots \le r_\ell \le [\pi_{i_0} n]} \Big(\sum_{k=1}^{\ell} \sum_{i=r_{k-1}}^{r_k-1} Y^*_{k,i} \Big) \Big] \to 0 .
\end{aligned} \tag{3.41}
$$

Finally, note that if $\ell_2(n) - \ell_1(n)$ is replaced by n in (3.36), (3.37), we get that

$$
\begin{aligned}
n^{-1/2}\Big[& \max_{0 \le r_1 \le \dots \le r_\ell \le [\pi_{i_0} n]} \Big(\sum_{k=1}^{\ell} \sum_{i=r_{k-1}}^{r_k-1} Y^*_{k,i} \Big) \\
& - \max_{0 \le r_1 \le \dots \le r_m \le [\pi_{i_0} n]} \Big(\sum_{k=1}^{m} \sum_{i=r_{k-1}}^{r_k-1} Y^*_{\ell_k,i} \Big) \Big] \xrightarrow{P} 0 .
\end{aligned}
$$

which coupled with (3.41) completes the proof of the lemma. □

In the following section, the limit behavior of the critical and supercritical compartments is established.

4. THE ASYMPTOTIC BEHAVIOR OF THE NORMALIZED PROCESS

The behavior of the normalized critical and supercritical compartments can now be established. In addition, it will be shown that the limit distribution of the normalized compartments is independent of the limit distribution of the subcritical compartments. The lemma needed to establish the asymptotic independence of the subcritical and the critical and supercritical compartments will be stated without proof, since it is straightforward.

LEMMA 4.1 Let $\{(\underset{\sim}{Y}_1(n), \underset{\sim}{Y}_2(n)), n=1,2,\ldots\}$ be a sequence of random vector couplets defined on a Markov chain $\{X_n\}$. Let $\underset{\sim}{Y}_1(n) \xrightarrow{P} \underset{\sim}{Y}_1$ as $n \to \infty$. Further, for all $m > 0$, let there exist a random vector sequence $\{\underset{\sim}{Y}_n^{(m)}, n = m, m+1, \ldots\}$ defined on the chain $\{X_n\}$ such that

1) $\underset{\sim}{Y}_n^{(m)} - \underset{\sim}{Y}_2(n) \xrightarrow{P} 0$, as $n \to \infty$, for all $m > 0$.

2) $P(\underset{\sim}{Y}_1(m) \le \underset{\sim}{x}, \underset{\sim}{Y}_n^{(m)} \le \underset{\sim}{y} | X_m) = P(\underset{\sim}{Y}_1(m) \le x | X_m) P(\underset{\sim}{Y}_n^{(m)} \le \underset{\sim}{y} | X_m)$, for $n \ge m$,

3) $P(\underset{\sim}{Y}_n^{(m)} \le \underset{\sim}{y} | X_m = i) \to P(\underset{\sim}{Y}_2 \le \underset{\sim}{y})$ as $n \to \infty$, for all $i \in J$, for all m, for all continuity points $\underset{\sim}{y}$ of $\underset{\sim}{Y}_2$. Then, for all continuity points $(\underset{\sim}{x}, \underset{\sim}{y})$ of the distribution $(\underset{\sim}{Y}_1, \underset{\sim}{Y}_2)$, we have

$$\lim_{n\to\infty} P(\underset{\sim}{Y}_1(n) \le \underset{\sim}{x}, \underset{\sim}{Y}_2(n) \le y) = P(\underset{\sim}{Y}_1 \le \underset{\sim}{x})\, P(\underset{\sim}{Y}_2 \le \underset{\sim}{y}).$$

We now establish the main theorem of this paper. In the k compartment model, let $(i_1, \ldots, i_m)$ be the indices for the subcritical compartments, and let $(j_1, \ldots, j_\ell)$ be the indices for the critical or supercritical compartments, so $m + \ell = k$. Further, for ease of notation we define for $1 \le i \le k$, μ_i by

$$\mu_i = \min_{0 \le j < i} (E_\pi V_j) - E_\pi V_i . \tag{4.1}$$

Then the following theorem holds.

<u>THEOREM 4.2</u> <u>If</u> $E_\pi|V_i| < \infty$, $E[\tilde{V}^2_{i,1}|X_0 = i_0) < \infty$ <u>for</u> $0 \le i \le k$, <u>then</u>

$$\lim_{n\to\infty} P(Z_{i_1}(n) \le x_{i_1}, \ldots, Z_{i_m}(n) \le x_{i_m}, Z_{j_1}(n) - n\mu_{j_1} \le x_{j_1} n^{1/2},$$

$$\ldots, Z_{j_\ell}(n) - n\mu_{j_\ell} \le x_{j_\ell} n^{1/2})$$

$$= P(Z_{i_1} \le x_{i_1}, \ldots, Z_{i_m} \le x_{i_m}) \cdot F(W_0(\cdot), \ldots, W_k(\cdot))$$

<u>for all continuity points</u> $(x_{i_1}, \ldots, x_{i_m})$ <u>of</u> $(Z_{i_1}, \ldots, Z_{i_m})$, <u>where</u> $W_0(\cdot), \ldots, W_k(\cdot)$ <u>is multivariate Brownian motion with</u> 0 <u>drift and appropriate variance matrix</u> σ^2, <u>and</u> $F(\cdot)$ <u>is some appropriate functional</u>.

<u>PROOF</u>. We will first find asymptotically equivalent expressions for the critical or supercritical compartments. Let compartment ℓ be either critical or supercritical. From (2.1) it can be shown that

$$\begin{aligned} Z_\ell(n) = &\min_{0 \le j_1 \le \ldots \le j_w \le n} [S_0(j_1) + (S_1(j_2) - S_1(j_1)) + \ldots + (S_w(n) - S_w(j_w))] \\ &- \sum_{j=w+1}^{\ell-1} Z_j(n) - \min_{0 \le j_1 \le \ldots \le j_\ell \le n} [S_0(j_1) + \ldots + (S_\ell(n) - S_\ell(j_\ell))] \end{aligned} \tag{4.2}$$

for w defined as $w = \max\{j < \ell : E_\pi V_j = \min_{0 \le i < \ell} E_\pi V_i\}$. By (Tollar [12]), we have that $n^{-1/2} Z_i(n) \xrightarrow{P} 0$, for $w < i < \ell$. Thus we need only consider

$$\min_{0 \le j_1 \le \dots \le j_w \le n} [S_0(j_1) + \dots + (S_w(n) - S_w(j_w))] \tag{4.3}$$

$$- \min_{0 \le j_1 \le \dots \le j_\ell \le n} [S_0(j_1) + \dots + (S_\ell(n) - S_\ell(j_\ell))]$$

$$= S_w(n) - \max_{0 \le j_1 \le \dots \le j_w \le n} \{ \sum_{m=0}^{w-1} ([S_w(j_{m+1}) - S_w(j_m)] - [S_m(j_{m+1}) - S_m(j_m)]) \}$$

$$- S_\ell(n) + \max_{0 \le j_1 \le \dots \le j_\ell \le n} \{ \sum_{m=0}^{\ell-1} ([S_\ell(j_{m+1}) - S_\ell(j_m)] - [S_m(j_{m+1}) - S_m(j_m)]) \} .$$

From the definition of w and the assumption that cell ℓ is either critical or supercritical, we know that $E_\pi V_m - E_\pi V_\ell \le 0$ for $0 \le m < \ell$, and $E_\pi V_m - E_\pi V_w \le 0$ for $0 \le m < w$. Also, it is well known (see Chung [3]) that

$$n^{-1/2}[S_m(n) - \sum_{i=1}^{[\pi_{i_0} n]} V^*_{m,i}] \xrightarrow{p} 0 . \tag{4.4}$$

Thus, from Theorem (3.3) and (4.3) we have

$$n^{-1/2}[Z_\ell(n) - \sum_{i=1}^{[\pi_{i_0} n]} (V^*_{w,i} - V^*_{\ell,i})$$

$$+ \max_{0 \le j_1 \le \dots \le j_r \le [\pi_{i_0} n]} \{ \sum_{m=1}^{r} \sum_{i=j_{m-1}+1}^{j_m} (V^*_{w,i} - V^*_{k_m,i}) \}$$

$$- \max_{0 \le j_1 \le \dots \le j_{r+1} \le [\pi_{i_0} n]} \{ \sum_{m=1}^{r} \sum_{i=j_{m-1}+1}^{j_m} (V^*_{\ell,i} - V^*_{k_m,i}) \tag{4.5}$$

$$+ \sum_{i=j_r+1}^{j_{r+1}} (V^*_{\ell,i} - V^*_{w,i}) \}] \xrightarrow{p} 0,$$

where $(k_1, \dots, k_r)$ are the indices less than w where $E_\pi V_{k_i} = E_\pi V_w$. As such,

$$n^{-1/2}[Z_{\ell}(n) - n\mu_{\ell} - \sum_{i=0}^{[\pi_{i_0} n]} \{V^*_{w,i} - V^*_{\ell,i} - \pi_{i_0}^{-1} \mu_{\ell}\}$$

$$+ \max_{0 \le j_1 \le \dots \le j_r \le [\pi_{i_0} n]} \left(\sum_{m=1}^{r} \sum_{i=j_{m-1}-1}^{j_m} V^*_{w,i} - V^*_{k_m,i} \right)$$

$$- \max_{0 \le j_1 \le \dots \le j_{r+1} \le [\pi_{i_0} n]} \left\{ \sum_{m=1}^{r} \sum_{i=j_{m-1}+1}^{j_m} (V^*_{\ell,i} - V^*_{k_m,i}) \right. \tag{4.6}$$

$$\left. + \sum_{i=j_r+1}^{j_{r+1}} (V^*_{\ell,i} - V^*_{w,i}) \right\}] \xrightarrow{P} 0 .$$

Note that in (4.6), if $E_\pi V_w > E_\pi V_\ell$, then from lemma 3.2 it follows that

$$n^{-1/2} \max_{0 \le j_1 \le \dots \le j_{r+1} \le [\pi_{i_0} n]} \left(\sum_{m=1}^{r} \sum_{i=j_{m-1}+1}^{jm} V^*_{\ell,i} - V^*_{k_m,i} \right) + \sum_{i=j_r+1}^{j_{r+1}} (V^*_{\ell,i} - V^*_{w,i})) \xrightarrow{P} 0 , \tag{4.7}$$

in which case, this term can be deleted from (4.6).

Thus, from (4.6) we have that

$$n^{-1/2}[(Z_{j_1}(n) - n\mu_{j_1}, Z_{j_2}(n) - n\mu_{j_2}, \dots, Z_{j_\ell}(n) - n\mu_{j_\ell})]$$

is asymptotically equivalent to a functional of $(Y_{0,n}(\cdot), \dots, Y_{k,n}(\cdot))$, where for $0 \le t \le 1$,

$$Y_{\ell,n}(t) = n^{-1/2} \sum_{i=1}^{[n\pi_{i_0} t]} (V^*_{\ell,i} - \pi_{i_0}^{-1} E_\pi V_\ell) . \tag{4.8}$$

To see this, one need only note that

$$n^{-1/2} \max_{0 \le j_1 \le \ldots \le j_r \le [\pi_{i_0} n]} \left(\sum_{m=1}^{r} \sum_{i=j_{m-1}+1}^{j_m} V^*_{w,i} - V^*_{k_m,i} \right)$$

$$= \sup_{0 \le s_1 \le \ldots \le s_r \le 1} [Y_{w,n}(s_r) - \sum_{m=1}^{r} (Y_{k_m,n}(s_m) - Y_{k_m,n}(s_{m-1}))] , \tag{4.9}$$

with similar relations for the other necessary terms of (4.6). Thus, for $(j_1, j_2, \ldots, j_\ell)$ the indices of the critical or supercritical cells, we have

$$n^{-1/2}\{(Z_{j_1}(n)-n\mu_{j_1},\ldots,Z_{j_\ell}(n)-n\mu_{j_\ell})\}-h(Y_{0,n}(\cdot),\ldots,Y_{k,n}(\cdot)) \xrightarrow{p} 0, \tag{4.10}$$

for $h(\cdot)$ an appropriate functional on $C^{k+1}[0,1]$. That $h(\cdot)$ is continuous is clear from (4.6), so we have by the uniform convergence of $(Y_{0,n}(1),\ldots Y_{k,n}(1))$ to multivariate Brownian motion (see Billingsley [2], Donsker [5]) that

$$h(Y_{0,n}(\cdot), \ldots, Y_{k,n}(\cdot)) \xrightarrow{d} h(W_0(\cdot), \ldots, W_k(\cdot)), \tag{4.11}$$

where $(W_0(\cdot), \ldots, W_k(\cdot))$ is a multivariate Brownian motion with drift 0 and variance matrix σ^2 given by

$$(\sigma'\sigma)_{ij} = \pi_{i_0} \operatorname{cov}(V^*_{i,1}, V^*_{j,1}) . \tag{4.12}$$

To complete the proof, we need only show the asymptotic independence of the critical and supercritical cells from the subcritical ones. This can be accomplished by appealing to Lemma 4.1. For $(i_1, \ldots, i_m)$ the subcritical indices, we know from (Tollar [12]) that

$$(\hat{Z}_{i_1}(n), \ldots, \hat{Z}_{i_m}(n)) \xrightarrow{a.s.} (Z_{i_1}, \ldots, Z_{i_m}). \tag{4.13}$$

Also, for $(j_1, \ldots, j_\ell)$ the critical or supercritical cells, let

$$\hat{Z}_{j_s}^{(m)}(n) = \min_{m \le r_1 \le \dots \le r_{j_s-1} \le n} \Big(\sum_{i=m+1}^{r_1} V_{j_s-1,i}(\hat{X}_i)$$

$$+ \sum_{i=r_1+1}^{r_2} V_{j_s-2,i}(\hat{X}_i) + \dots + \sum_{i=r_{j_s-1}+1}^{n} V_{0,i}(\hat{X}_i)\Big) \tag{4.14}$$

$$- \min_{m \le r_1 \le \dots \le r_{j_s} \le n} \Big(\sum_{i=m+1}^{r_1} V_{j_s,i}(\hat{X}_i) + \dots + \sum_{i=r_{j_s}+1}^{n} V_{0,i}(\hat{X}_i)\Big).$$

To apply Lemma 4.1, let

$$\underset{\sim}{Y}_1(n) = (\hat{Z}_{i_1}(n), \dots, \hat{Z}_{i_m}(n)),$$

$$\underset{\sim}{Y}_2(n) = n^{-1/2}(\hat{Z}_{j_1}(n) - n\mu_{j_1}, \dots, \hat{Z}_{j_\ell}(n) - n\mu_{j_\ell}))$$

$$\underset{\sim}{Y}_n^{(m)} = n^{-1/2}(\hat{Z}_{j_1}^{(m)}(n) - n\mu_{j_1}, \dots, \hat{Z}_{j_\ell}^{(m)}(n) - n\mu_{j_\ell})).$$

Clearly the conditions of the lemma are met, which yields the sought after independence. This completes the proof of the theorem. □

To highlight the nature of the functional cited in Theorem 4.2, we will state without proof the marginal asymptotic behavior of any critical or supercritical cell as a corollary of Theorem 4.1, as it follows directly from (4.6) and (4.7).

<u>COROLLARY 1</u>. <u>For a fixed</u> ℓ, <u>let</u> $w = \max\{j < \ell: E_\pi V_j = \min_{0 \le j < \ell} E_\pi V_i\}$, <u>and</u> $(k_1, k_2, \dots, k_r)$ <u>be the indices such that</u> $k_i < w$ <u>and</u> $E_\pi V_{k_i} = E_\pi V_w$. <u>Then if</u>

1) compartment ℓ is supercritical,

$$n^{-1/2}\{Z_\ell(n) - n(E_\pi V_w - E_\pi V_\ell)\} \xrightarrow{d} W_w(1) - W_\ell(1)$$
$$- \sup_{0 \le s_1 \le \dots \le s_r \le 1} (W_w(s_r) - \sum_{m=1}^{r} (W_{k_m}(s_m) - W_{k_m}(s_{m-1})));$$

or if 2) compartment ℓ is critical,

$$n^{-1/2} Z_\ell(n) \xrightarrow{d} W_w(1) - W_\ell(1) - \sup_{0 \le s_1 \le \dots \le s_r \le 1} (W_w(s_r) - \sum_{m=1}^{r} (W_{k_m}(s_m) - W_{k_m}(s_{m-1})))$$
$$+ \sup_{0 \le s_1 \le \dots \le s_{r+1} \le 1} (W_\ell(s_{r+1}) - \sum_{m=1}^{r} (W_{k_m}(s_m) - W_{k_m}(s_{m-1})) - (W_w(s_{r+1}) - W_w(s_r))),$$

where $(W_0(\cdot), \dots, W_k(\cdot))$ is multivariate Brownian motion with drift 0 and σ^2 given by (4.12).

The expressions for $h(\cdot)$ given in Theorem 4.2 and Corollary 1 can obviously be improved upon for the various specific arrangements of critical, supercritical, and subcritical cells. Some of the limit distributions for certain arrangements of critical and supercritical cells have densities that can be expressed in integral form, but it should be noted that many of the limit distributions seem to not have such simplified forms. As such, the expressions in Corollary 1 cannot be noticeably simplified to expressions that do not use functionals of Brownian motion.

5. CONCLUSION

There are several directions of further research left unanswered in this paper. While the asymptotic behavior of the critical and supercritical models were given in Corollary 1, nothing about the limiting distribution for the subcritical cells other than existance was specified. Any characterization of this limit, however, seems to be extremely difficult. For the single cell model, results were obtained by Puri [12], but the techniques there do not seem to generalize to the present case.

Results for more general flow structure than the one-way flow used in this paper seem to require different techniques than used above. Unlike the one-way flow model, there appears to be no closed form expression for $Z_i(n)$ in the more general framework.

The model could be extended to continuous time by considering an underlying semi-Markov process instead of a Markov chain. Results in this area are presently under preparation.

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